Signed edge k-subdomination numbers in graphs

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Abstract

The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum_{x\in N[e]} f(x) \geq 1$ for at least k edges e of G, then f is called a signed edge k-subdominating function of G. The minimum of the values $\sum_{e\in E(G)} f(e)$, taken over all signed edge k-subdominating function f of G, is called the signed edge k-subdomination number of G and is denoted by $\gamma'_{ks}(G)$. In this note we initiate the study of the signed edge k-subdomination in graphs and present some (sharp) bounds for this parameter.

Keywords: signed edge dominating function; signed domination number; signed edge k-subdominating function; signed edge k-subdomination number

1 Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [8] for terminology and notation which are not defined here. The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. If $S = N_G[e]$ for some $e \in E$, then we denote $f(S)$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e\in E(v)} f(e)$, where $E(v)$ is the set of all edges incident to vertex v. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge k-subdominating function* (SEkSDF) of G, if $f[e] \geq 1$ for at least k edges e

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of G. The minimum of the values $f(E(G))$, taken over all signed edge k-subdominating functions f of G, is called the *signed edge k-subdomination number* of G and is denoted by $\gamma'_{ks}(G)$. The signed edge k-subdominating function f of G with $f(E(G)) = \gamma'_{ks}(G)$ is called $\gamma'_{ks}(G)$-*function*. For any signed edge k-subdominating function f of G we define $P = \{e \in E(G) \mid f(e) = 1\}$, $M = \{e \in E(G) \mid f(e) = -1\}$ and $X = \{e \in E(G) \mid f[e] \geq 1\}$.

If $k = m$, where m is the size of a graph, then the signed edge k-subdomination number is called the *signed edge domination number*. The signed edge domination number was introduced by Xu in [9] and denoted by $\gamma'_s(G)$. This parameter has been studied by several authors [4, 5, 6, 9, 10, 12, 13, 14].

If $k = \lceil m/2 \rceil$, then the signed edge k-subdomination number is called the *signed edge majority domination number*. This parameter was introduced by Karami et al. in [7] and denoted by $\gamma'_{sm}(G)$.

An opinion function on a graph G is a function $f : V(G) \longrightarrow \{-1, 1\}$. By the vote of a vertex v we mean $\sum_{w\in N[v]} f(w)$. A *k-subdominating* function [1] of a graph G is an opinion function for which the votes of at least k vertices are positive. The *k-subdomination* number of G, denoted by $\gamma_{ks}(G)$, is the minimum of the values of $\sum_{v\in V(G)} f(v)$, taken over all k-subdominating functions f of G. The k-subdominating function f of G with $f(V(G)) = \gamma_{ks}(G)$ is called $\gamma_{ks}(G)$-*function*.

The following table shows the notation introduced above.

For $e \in E(G),\ N_G[e] = N_G(e) \cup \{e\}$
For $S \subseteq E(G),\ f(S) = \sum_{e\in S} f(e)$
For $e \in E(G),\ f[e] = \sum_{e'\in N[e]} f(e')$
For $S = N_G[e],\ f(S) = f[e]$
For $v \in V(G),\ E(v) =$ the set of all edges incident to vertex v
For $v \in V(G),\ f(v) = \sum_{e\in E(v)} f(e)$
$P = \{e \in E(G) \mid f(e) = 1\}$
$M = \{e \in E(G) \mid f(e) = -1\}$
$X = \{e \in E(G) \mid f[e] \geq 1\}$
$\gamma'_s(G) =$ the signed edge domination number of G
$\gamma'_{sm}(G) =$ the signed edge majority domination number of G
$\gamma'_{ks}(G) =$ the signed edge k-subdomination number of G
$\gamma_{ks}(G) =$ the k-subdomination number of G

Table of notation

In this note we initiate the study of the signed edge k-subdomination in graphs and present some (sharp) bounds for this parameter. Note that the signed edge k-subdomination number is a generalization of the signed edge domination number introduced by Xu in [9] and the signed edge majority domination number introduced by Karami et al. in [5]. Here are some well-known results on $\gamma'_s(G)$, $\gamma'_{sm}(G)$ and $\gamma_{ks}(G)$.

Theorem A. [4]) For any tree T of order $n \geq 2$, $\gamma'_s(T) \geq 1$ with equality if and only if T has no vertex of even degree and $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ for every vertex v, where $\ell(v)$ denotes the number of pendant edges at vertex v. In addition, if $\gamma'_s(T) = 1$ and f is a $\gamma'_s(T)$-function, then $f(v) = 1$ for every vertex of degree greater than one.

Theorem B. ([11]) Let G be a graph with $\delta(G) \geq 1$. Then $\gamma'_s(G) \geq |V(G)| - |E(G)|$ and this bound is sharp.

Define $\mathcal{G}_0$ to be the collection of all simple connected graphs of order $n \geq 2$ in which the degree of each vertex is odd and $\ell(v) \geq (\deg(v) - 1)/2$ for every vertex v.

Theorem C. ([5]) Let G be a simple connected graph of order $n \geq 2$ and size m. Then $\gamma'_s(G) = n - m$ if and only if $G \in \mathcal{G}_0$. Furthermore, if $\gamma'_s(G) = n - m$ and f is a $\gamma'_s(G)$-function, then

1. $f(e) = 1$ for each non-pendant edge $e \in E(G)$;
2. $f(v) = 1$ for each vertex v of degree greater than 1;
3. $f[e] = 1$ for each edge $e \in E(G)$.

Theorem D. ([10]) For any positive integer m, define

$$\Psi(m) = \min\{\gamma'_s(G) \mid G \text{ is a graph of size } m\}.$$

Then

$$\Psi(m) = 2\lceil \frac{1}{3} \lceil \frac{\sqrt{24m+25} + 6m + 5}{6} \rceil \rceil - m.$$

Theorem E. ([7]) Let Ψ be as in Theorem D. Then

1. Ψ is an increasing function,
2. $m \geq \Psi(m)$ for every positive integer m, and
3. $\Psi(a) + \Psi(b) \geq \Psi(a + b)$ for each pair of positive integers a and b.

Theorem F. ([7]) Let G be a simple graph of order $n \geq 3$ and size m. Then

$$\gamma'_{sm}(G) \geq \Psi(\lceil \frac{m}{2} \rceil) - \lfloor \frac{m}{2} \rfloor.$$

Furthermore, this bound is sharp.

Theorem G. ([1]) For $n \geq 2$ and $1 \leq k \leq n$, $\gamma_{ks}(P_n) = 2\lfloor (2k + 4)/3 \rfloor - n$.

Theorem H. ([2]) If $n \geq 3$ and $1 \leq k \leq n - 1$, then

$$\gamma_{ks}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n - 1 \text{ and } k \equiv 1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

Theorem I. ([1]) For any connected graph G of order n and any positive integer $k \leq \lceil \frac{n}{2} \rceil$,

$$\gamma_{ks}(G) \leq \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem J. ([3]) For any connected graph G of order n and any positive integer k with $\frac{n}{2} < k \leq n$,

$$\gamma_{ks}(G) \leq 2\lceil \frac{k}{n - k + 1} \rceil (n - k + 1) - n.$$

The proof of the following theorem is straightforward and therefore omitted.

Theorem 1. For any graph G of order $n \geq 2$ which has no isolates,

$$\gamma'_{ks}(G) = \gamma_{ks}(L(G)).$$

Theorems 1, G, H, I and J lead to:

Corollary 2. For $n \geq 3$ and $1 \leq k \leq n-1$,

$$\gamma'_{ks}(P_n) = 2\lfloor (2k+4)/3 \rfloor - n + 1.$$

Corollary 3. For $n \geq 3$ and $1 \leq k \leq n-1$,

$$\gamma'_{ks}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n-1 \text{ and } k \equiv 1 \pmod 3 \\ \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

Corollary 4. For any connected graph G of size m and positive integer $k \leq \lceil \frac{m}{2} \rceil$,

$$\gamma'_{ks}(G) \leq \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even.} \end{cases}$$

These bounds are sharp for stars.

Corollary 5. For any connected graph G of size m and any positive integer k with $\frac{m}{2} < k \leq m$,

$$\gamma'_{ks}(G) \leq 2\lceil \frac{k}{m-k+1} \rceil (m-k+1) - m.$$

2 Lower bounds on the SEkSDNs of graphs

In this section we first generalize Theorem F to the signed edge k-subdomination number. Then we present a lower bound for $\gamma'_{ks}(G)$ in terms of the size, the minimum degree and the maximum degree of G. We find a sharp lower bound for $\gamma'_{ks}(T)$ in terms of k, the order and the size of tree T. This generalizes the lower bound given in Theorem A to the SEkSDNs of trees. Finally, we find a lower bound for $\gamma'_{ks}(G)$ in terms of k, the order and the size of G, generalizing Theorem B. We show that this bound is sharp for $k \geq 7$ and odd.

Theorem 6. For any simple graph G of order $n \geq 3$ and size m,

$$\gamma'_{ks}(G) \geq \Psi(k) - (m-k),$$

where $1 \leq k \leq m$. Furthermore, this bound is sharp.

Proof. The statement holds for all simple graphs of size $m = 1, 2, 3$. Now assume $m \geq 4$. Let, to the contrary, G be a simple graph of size $m \geq 4$ such that $\gamma'_{ks}(G) < \Psi(k) - (m-k)$. Choose such a graph G with as few edges as possible for which $\omega(G) + |T(G)|$ is maximum, where $\omega(G)$ denotes the number of components of G and $T(G) = \{u \in V(G) \mid \deg(u) \leq 2\}$. Without loss of generality we may assume G has no isolated vertices. Let f be a $\gamma'_{ks}(G)$-function. Let $G_1, \ldots, G_{\omega(G)}$ be the connected components of G. If $G_i \simeq K_2$ for each $1 \leq i \leq \omega(G)$, then obviously

$$\gamma'_{ks}(G) = k - (m-k) \geq \Psi(k) - (m-k).$$

Let G have a connected component H of size at least 2. First we prove that $E(H) \subseteq X$. Hence, $f|_H$ is actually a γ'_s-function on H for every connected component H with $|E(H)| \geq 3$ of G. Then we use a simple counting argument and Lemma E to complete the proof.

Claim 1. $E(H) \cap M \subseteq X$.

Let $e \in E(H) \cap M$. Suppose that, to the contrary, $e \notin X$. Assume G' is obtained from $G - e$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G') \setminus \{u_0v_0\}$. Obviously, g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G. Thus $e \in X$.

Claim 2. For every non-pendant edge $e = uv \in E(H) \cap M$ we have $\deg(u) = \deg(v) = 2$.

If $f(u) \geq 1$ (the case $f(v) \geq 1$ is similar) and G' is obtained from $G - e$ by adding a pendant edge uv', then obviously $g : E(G') \longrightarrow \{-1, 1\}$, which is defined by $g(uv') = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$, is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G. Hence, $f(u) = f(v) = 0$. Therefore $\deg(u)$ and $\deg(v)$ are even. Let $\deg(u) \geq 4$ (the case $\deg(v) \geq 4$ is similar). Then there is a $+1$ edge $e' = uw$ at u. Assume G' is obtained from $G - \{e, e'\}$ by adding a new vertex z and two new edges vz and wz. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(vz) = -1$, $g(wz) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, e'\}$. Then g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Hence, $\deg(u) = \deg(v) = 2$.

Claim 3. Let $e = uv \in E(H) \cap M$ be a non-pendant edge and $uu', vv' \in E(G)$. Then $uu', vv' \in X$.

Let, to the contrary, $uu' \notin X$ (the case $vv' \notin X$ is similar). Since $e \in X$, $f(uu') = f(vv') = 1$. Suppose that $\deg(u') = 1$ and G' is obtained from $G - \{e, uu'\}$ by adding a pendant edge vv_1 and a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(vv_1) = -1$, $g(u_0v_0) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu'\}$. Then g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $\deg(u') \geq 2$. Similarly, we can see that $\deg(v') \geq 2$.

First let $u' = v'$. Since $uu' \notin X$, we have $vv' \notin X$. Suppose that there exists a -1 pendant edge $u'z$ at u'. By Claim 1, $u'z \in X$, which implies that $f(u') \geq 1$. Let G' be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$. Obviously, g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore, there is no -1 pendant edge at $u' = v'$. If there exists a -1 non-pendant edge at u', then an argument similar to that described in Claim 2 shows that $\deg(u') = 2$, a contradiction. Thus every edge at u' is a $+1$ edge. This forces $uu' \in X$, a contradiction.

Now let $u' \neq v'$. Since we have assumed $uu' \notin X$, it follows that $f(u') \leq 1$. If there is a -1 pendant edge $u'w$ at u', then by Claim 1 we have $u'w \in X$ and hence, $f(u') = f[u'w] \geq 1$. If there is a -1 non-pendant edge at u', then $\deg(u') = 2$ by Claim 2 and hence, $f(u') = 0$. It follows that $f(u') = 0, 1$.

When $f(u') = 1$, define G' to be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$ is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $f(u') = 0$ and hence, there exists a -1 edge $u'u''$ at u'. If $\deg(u'') = 1$, define G' to be the graph obtained from $G - \{u'u''\}$ by adding a new component u_0v_0. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{u'u''\}$ is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| >$

$\omega(G)+|T(G)|$, a contradiction. Hence, $\deg(u'')=2$ (see Claim 2). Let G' be obtained from $G-\{e,uu',u'u''\}$ by adding a new component u_0v_0 and two new edges $u''z, zv$. Then $g: E(G')\longrightarrow\{-1,1\}$ defined by $g(u_0v_0)=-1$, $g(u''z)=-1$, $g(zv)=1$ and $g(x)=f(x)$ if $x\in E(G)\backslash\{e,uu',u'u''\}$ is an SEkSDF of G' with $g(E(G'))=f(E(G))$ and $\omega(G')+|T(G')|>\omega(G)+|T(G)|$, a contradiction. Therefore $uu'\in X$, a contradiction.

Claim 4. $E(H)\cap P\subseteq X$.

Let $e=uv\in E(H)\cap P$. If there is a -1 non-pendant edge at u or at v, then by Claim 3 we have $e\in X$. If there exists a -1 pendant edge e' at u, then $e'\in X$ by Claim 1 and hence, $f(u)=f[e']\geq 1$. If all the edges at u are $+1$ edges, then $f(u)\geq 1$. Similarly, if there is no -1 non-pendant edge at v, we see that $f(v)\geq 1$. Hence, $e\in X$.

Let $G_1,\ldots,G_s$ be the connected components of G for which $E(G_i)\subseteq X$. Thus, $f|_{G_i}$ is a γ'_s-function on G_i for each $1\leq i\leq s$. Now by Claims 1 and 3, $X\cap[\cup_{i=s+1}^{w(G)}E(G_i)]=\emptyset$.

Let $|E(G_i)|=m_i$ for each $1\leq i\leq w(G)$. Then $|X|=\sum_{i=1}^{s}m_i\geq k$ and $\sum_{i=s+1}^{w(G)}m_i\leq m-k$. Then by Lemma E,

$$\begin{aligned}\gamma'_{ks}(G) &= \textstyle\sum_{i=1}^{s}\gamma'_s(G_i)-\sum_{i=s+1}^{w(G)}m_i\\ &\geq \textstyle\sum_{i=1}^{s}\Psi(m_i)-\sum_{i=s+1}^{w(G)}m_i\\ &\geq \textstyle\Psi(\sum_{i=1}^{s}m_i)-\sum_{i=s+1}^{w(G)}m_i\\ &\geq \Psi(k)-(m-k)\end{aligned}$$

In order to prove that the lower bound is sharp, let H_1 be a graph of size k with $\gamma'_s(H)=\Psi(k)$ (see [10]) and let H_2 be a graph of size $m-k$ such that $V(H_1)\cap V(H_2)=\emptyset$. Suppose $G=H_1\cup H_2$ and f is a $\gamma'_s(H_1)$-function. Then $g: E(G)\longrightarrow\{-1,1\}$ defined by $g(e)=f(e)$ if $e\in E(H_1)$ and $g(e)=-1$ if $e\in E(H_2)$, is an SEkSDF of G with $g(E(G))=\Psi(k)-(m-k)$. This completes the proof. □

Theorem 7. Let G be a simple graph of size m, minimum degree δ, maximum degree Δ and no isolates. Then

$$\gamma'_{ks}(G)\geq\frac{2k\delta}{2\Delta-1}-m.$$

Proof. Let $(d_1,\ldots,d_n)$ be the degree sequence of G where $d_1\leq d_2\leq\ldots\leq d_n$. Assume g is a $\gamma'_{ks}(G)$-function of G and let $g(e)\geq 1$ for k distinct edges e in $\{e_{j_1}=u_{j_1}v_{j_1},\ldots,e_{j_k}=u_{j_k}v_{j_k}\}$. Define $f: E(G)\longrightarrow\{0,1\}$ by $f(e)=(g(e)+1)/2$ for each $e\in E(G)$. We have

$$\begin{aligned}\textstyle\sum_{i=1}^{k}f(N_G[e_{j_i}]) &= \textstyle\sum_{i=1}^{k}\frac{g(N_G[e_{j_i}])+\deg(u_{j_i})+\deg(v_{j_i})-1}{2}\\ &\geq \textstyle\sum_{i=1}^{k}\frac{\deg(u_{j_i})+\deg(v_{j_i})}{2}\\ &\geq k\delta.\end{aligned}\tag{1}$$

On the other hand,

$$\begin{aligned}\textstyle\sum_{i=1}^{k}f(N_G[e_{j_i}]) &\leq \textstyle\sum_{e\in E}f(N_G[e])=\sum_{e=uv\in E}(\deg(u)+\deg(v)-1)f(e)\\ &\leq \textstyle\sum_{e\in E}(2\Delta-1)f(e)\\ &= (2\Delta-1)f(E(G)).\end{aligned}\tag{2}$$

By (1) and (2), $f(E(G))\geq\dfrac{k\delta}{2\Delta-1}$. Since $g(E(G))=2f(E(G))-m$,

$$\gamma'_{ks}(G) = g(E(G)) \geq \frac{2k\delta}{2\Delta - 1} - m,$$

as desired. □

As an immediate consequence of Theorem 7 we have:

Corollary 8. For every r-regular ($r \geq 1$) graph G of size m, $\gamma'_{ks}(G) \geq \dfrac{2rk}{2r-1} - m$. Furthermore, this bound is sharp when $r = 1$.

Now we prove that for any tree of size $m \geq 2$ and any integer $1 \leq k \leq m-1$, $\gamma'_{ks}(T) \geq k - m + 2$. This generalizes the lower bound for signed edge domination numbers for trees given in Theorem A to the signed edge k-subdomination numbers for trees.

Theorem 9. If T is a tree of size $m \geq 2$ and k is an integer, $1 \leq k \leq m-1$, then $\gamma'_{ks}(T) \geq 2\lceil k/2 \rceil - m + 2$. Furthermore, these bounds are sharp for each value of k.

Proof. First, by induction on m, we prove that $\gamma'_{ks}(T) \geq k - m + 2$, where $1 \leq k \leq m-1$. The statement holds for all trees of size $m = 2, 3, 4$. Assume T is an arbitrary tree of size $m \geq 5$ and that the statement holds for all trees of smaller sizes. Let f be a γ'_{ks}-function for T. If $M = \{e \in E(T) \mid f(e) = -1\} = \emptyset$, then obviously the theorem is true. Let $M \neq \emptyset$.

Case 1. There is a non-pendant edge $e = uv \in E(T)$ for which $f(e) = -1$.
Let T_1 and T_2 be the connected components of $T - e$ with $u \in T_1$. Then, $\gamma'_{ks}(T) = f(E(T_1)) - 1 + f(E(T_2))$. We consider two subcases.

Subcase 1.1 For $i = 1, 2$, $X \cap E(T_i) \neq \emptyset$, where $X = \{e \in E(T) \mid f[e] \geq 1\}$. Let $|X \cap E(T_1)| = k_1$ and $|X \cap E(T_2)| = k_2$. Then for $i = 1, 2$, the function f, restricted to T_i is an SEk_iSDF for T_i. Hence, $\gamma'_{k_is}(T_i) \leq f(E(T_i))$ for $i = 1, 2$. First let $E(T_i) \subseteq X$ for $i = 1, 2$. Then since $k \leq m-1$, it follows that $e \notin X$ and by Theorem A,

$$\gamma'_{ks}(T) = f(E(T)) = f(E(T_1)) - 1 + f(E(T_2)) \geq 1 - 1 + 1 \geq k - m + 2.$$

Now without loss of generality we assume $E(T_1) \not\subseteq X$. If $E(T_2) \subseteq X$, then by the inductive hypothesis and Theorem A, $\gamma'_{k_1s}(T_1) \geq k_1 - |E(T_1)| + 2$ and $\gamma'_{k_2s}(T_2) \geq 1$. If $e \notin X$, then obviously $\gamma'_{ks}(T) \geq k - m + 3$. If $e \in X$, then

$$\gamma'_{ks}(T) = f(E(T)) = f(E(T_1)) - 1 + f(E(T_2)) \geq k_1 - |E(T_1)| + 2 - 1 + 1 \geq k - m + 2.$$

The case $E(T_2) \not\subseteq X$ is similar.

Subcase 1.2 $X \cap E(T_1) = \emptyset$ (the case $X \cap E(T_2) = \emptyset$ is similar). First let $e \notin X$. Then $|X \cap E(T_2)| = k_2 \geq k$. We claim that f assigns -1 to all edges of T_1. If $E(T_1) \cap P \neq \emptyset$, where $P = \{e \in E(T) \mid f(e) = 1\}$, then we define $g : E(T) \longrightarrow \{-1, +1\}$ by $g(e) = -1$ if $e \in E(T_1)$ and $g(e) = f(e)$ if $e \in E(T) \setminus E(T_1)$. Then g is a SEkSDF of T of weight less than f, a contradiction. This proves our claim. Now by the inductive hypothesis on $T' = T_2 + uv$ we have

$$\gamma'_{ks}(T) = f(E(T_1)) + f(E(T')) \geq -|E(T_1)| + k_2 - |E(T')| + 2 \geq k - m + 2.$$

Now assume $e \in X$. First let $f(v) \geq 2$. Then f, restricted to T', is an SEkSDF for T'. If $k = |E(T')|$, since $f(v) \geq 2$, by Theorem A,

$$\gamma'_{ks}(T) \geq f(E(T')) - |E(T_1)| \geq (k - |E(T')|) + 2 - |E(T_1)| = k - m + 2.$$

If $k < |E(T')|$, by the inductive hypothesis,

$$\gamma'_{ks}(T) \geq f(E(T')) - |E(T_1)| \geq k - |E(T')| + 2 - (|E(T_1)|) \geq k - m + 2.$$

Now suppose that $f(v) = 1$. Then f, restricted to T', is an SEkSDF for T' and there exists at least one $+1$ edge at u in T_1 (note that $e \in X$). By Theorem A or the inductive hypothesis

$$\gamma'_{ks}(T) \geq f(E(T')) - |E(T_1)| + 2 \geq (k - |E(T')| + 1) - |E(T_1)| + 2 = k - m + 3.$$

Finally, let $f(v) \leq 0$. Then either there exists an edge at v not in X or there exists a vertex $w \in V(T_2)$ other than v for which $f(w) \geq 2$. Since $e \in X$, $f(u) \geq -f(v)$, hence in T_1 there are at least $-f(v) + 1$, $+1$ edges at u. Assume $E(v) \cap M = \{uv, vv_1, \ldots, vv_s\}$ and $E(u) \cap P = \{uu_1, uu_2, \ldots, uu_r\}$, where $r \geq s + 1$. Define $g : E(T) \longrightarrow \{-1, +1\}$ by $g(uv) = g(vv_i) = 1$, $g(uu_{s+1}) = g(uu_i) = -1$, for $i = 1, \ldots, s$ and $g(e) = f(e)$ if $e \in E(T) \setminus \{uu_{s+1}, uv, vv_i, uu_i \mid 1 \leq i \leq s\}$. Obviously, g is a $\gamma'_{ks}(T)$-function and $g(v) \geq 1$. Now by an argument similar to that described above (for the cases $f(v) \geq 2$ or $f(v) = 1$) we have $\gamma'_{ks}(T) = g(E(T)) \geq k - m + 2$.

Case 2. The only edges e for which $f(e) = -1$ are pendant edges.
First suppose that there exists a pendant edge $e = uv \in M \cap X^c$ with $\deg(u) = 1$. Then f, restricted to $T - u$, is an SEkSDF for $T - u$. If $k \leq m - 2$, then by the inductive hypothesis

$$\gamma'_{ks}(T) = f(E(T - u)) - 1 \geq k - (m - 1) + 2 - 1 = k - m + 2.$$

If $k = m - 1$, then obviously there exists a vertex $w \in V(T - \{u, v\})$ such that $f(w) \geq 2$. Now the result follows by Theorem A. Thus we may assume $M \subseteq X$. This implies that for each non leaf vertex $v \in V(T)$, $f(v) \geq 1$. Thus f is an SEDF for T and the result follows by Theorem A. This proves the first statement.

If k is odd, then $2|P| \neq k + 2$ and so $\gamma'_{ks}(G) = 2|P| - m \neq k - m + 2$. Hence, $\gamma'_{ks}(T) \geq k - m + 3$.

To prove sharpness, let T be the tree obtained from the star $K_{1,k+1}$, with vertex set $\{v, v_1, \ldots, v_{k+1}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k + 1\}$, by adding $m - k - 1$ pendant edges $v_1u_1, \ldots, v_1u_{m-k-1}$. Obviously, $\gamma'_{ks}(T) = k - m + 2$ for k even and $\gamma'_{ks}(T) = k - m + 3$ for k odd. This completes the proof. □

Now we find a lower bound for the SEkSDNs of simple connected graphs. This is a generalization of Theorem B.

Theorem 10. Let G be a simple connected graph of order $n \geq 3$, size m and $1 \leq k \leq m - 1$. Then

$$\gamma'_{ks}(G) \geq n + k + 1 - 2m.$$

Furthermore, the bound is sharp for each odd $k \geq 7$.

Proof. The proof is by induction on m. Obviously, the statement is true for $m = 2, 3$. Assume the statement is true for all simple connected graphs of size less than m, where $m \geq 4$. Let G be a simple connected graph of size m and let f be a $\gamma'_{ks}(G)$-function. we distinguish two cases.

Case 1 There is a non-pendant edge $e = uv \in E(G)$ for which $f(e) = -1$.
First let e not be a bridge. If $e \notin X = \{e \in E(T) \mid f[e] \geq 1\}$, then f is an SEkSDF of $G - e$. If $k \leq m - 2$, then by the inductive hypothesis

$$\gamma'_{ks}(G) = f(E(G)) = f(E(G-e)) - 1 \geq n + k + 1 - 2(m-1) - 1 = n + k + 2 - 2m.$$

If $k = m - 1$, then f is an SEDF of $G - e$ and by Theorem B we have

$$\gamma'_{ks}(G) = f(E(G)) = f(E(G-e)) - 1 \geq n - (m-1) - 1 = n - m = n + k + 1 - 2m.$$

Let $e \in X$. If f, restricted to $G - e$, is an SEDF of $G - e$, then the result follows as above. Otherwise f, restricted to $G - e$, is an SE(k-1)SDF of $G - e$ and the result follows by the inductive hypothesis.

Assume e is a bridge and G_1 and G_2 are the connected components of $G - e$. An argument similar to that described in the proof of Case 1 of Theorem 9 shows that $\gamma'_{ks}(G) \geq n + k + 1 - 2m$. Note that for this case we apply Theorem C instead of Theorem A.

Case 2 The only edges e for which $f(e) = -1$ are pendant edges.
First suppose that there exists a pendant edge $e = uv \in M \cap X^c$ with $\deg(u) = 1$. Then the result follows by an argument similar to that described in Case 2 of the proof of Theorem 9 and Theorem C. Now assume $M \subseteq X$. If G is a tree, then the statement is true by Theorem 9. Now let G have a cycle C. By assumption $f(e) = 1$ for every $e \in E(C)$. First assume $f(e') = 1$ for every edge e' with an end-vertex in $V(C)$. Then $f|_{G-e}$ is an SE(k-1)SDF of $G - e$ for every $e \in E(C)$. So we have

$$f(E(G)) = f|_{G-e}(E(G-e)) + 1 \geq n + (k-1) + 1 - 2(m-1) + 1 = n + k - 2m + 3.$$

Now assume there is an edge $e' = uu'$ with $f(e') = -1$ and $u \in V(C)$. Hence e' is a pendant edge by assumption. Let $e = uv \in E(C)$. If for every edge e'' at v, $f(e'') = 1$, then f restricted to $G_1 = G - \{e, e'\}$ is an SE(k-2)SDF of G_1. So we have

$$f(E(G)) = f|_{G_1}(E(G_1)) \geq (n-1) + (k-2) + 1 - 2(m-2) \geq n + k + 2 - 2m.$$

Finally, let there be an edge e'' at v with $f(e'') = -1$. Note that by assumption e'' is a pendant edge. Hence, f restricted to $G_2 = G - \{e, e', e''\}$ is an SE(k-3)SDF of G_2. So we have

$$f(E(G)) = f|_{G_2}(E(G_2)) - 1 \geq (n-2) + (k-3) + 1 - 2(m-3) - 1 \geq n + k + 1 - 2m.$$

To prove sharpness, we consider two cases.

- $k \geq 7$ is odd and $k = m - 1$. Let G be obtained from star $K_{1,k-3}$ with vertex set$\{v, v_1, \ldots, v_{k-3}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-3\}$ by adding three pendant edges $v_1v'_1, v_2v'_2, v_3v'_3$ and an edge v_1v_2. Define $f : V(G) \longrightarrow \{-1, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an SEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.

- $k \geq 7$ is odd and $k \leq m-2$. Let G be obtained from star $K_{1,k-2}$ with vertex set $\{v, v_1, \ldots, v_{k-2}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-2\}$ by adding pendant edges v_1v_1', v_2v_2', $v_3v_3^j$ for $j = 1, 2, \ldots, m-k-1$ and v_1v_2. Define $f : V(G) \longrightarrow \{-1, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an SEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.

This completes the proof. □

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