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Signed (b,k)-Edge Covers in Graphs

A. N. Ghameshlou[1], Abdollah Khodkar[2], R. Saei[3], S. M. Sheikholeslami*[3]
[1]*Department of Mathematics, University of Mazandaran Babolsar, I.R., Iran*
[2]*Department of Mathematics, University of West Georgia, Carrollton, USA*
[3]*Department of Mathematics, Azarbaijan University of Tarbiat MoallemTabriz, I.R., Iran*
Email: akhodkar@westga.edu, s.m.sheikholeslami@azaruniv.edu

Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let G have at least k vertices of degree at least b, where k and b are positive integers. A function $f: E(G) \rightarrow \{-1,1\}$ is said to be a signed (b,k)-edge cover of G if $\sum_{e \in E(v)} f(e) \geq b$ for at least k vertices v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. The value $\min \sum_{e \in E(G)} f(e)$, taking over all signed (b,k)-edge covers f of G is called the signed (b,k)-edge cover number of G and denoted by $\rho'_{b,k}(G)$. In this paper we give some bounds on the signed (b,k)-edge cover number of graphs.

Keywords: Signed Star Dominating Function, Signed Star Domination Number, Signed (b,k)-edge Cover, Signed (b,k)-edge Cover Number

1. Introduction

Structural and algorithmic aspects of covering vertices by edges have been extensively studied in graph theory. An *edge cover* of a graph G is a set C of edges of G such that each vertex of G is incident to at least one edge of C. Let b be a fixed positive integer. A *b-edge cover* of a graph G is a set C of edges of G such that each vertex of G is incident to at least b edges of C. Note that a *b*-edge cover of G corresponds to a spanning subgraph of G with minimum degree at least b. Edge covers of bipartite graphs were studied by König [1] and Rado [2], and of general graphs by Gallai [3] and Norman and Rabin [4], and *b*-edge covers were studied by Gallai [3]. For an excellent survey of results on edge covers and *b*-edge covers, see Schrijver [5].

We consider a variant of the standard edge cover problem. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [6] for terminology and notation which are not defined here and consider only simple graphs without isolated vertices. For every nonempty subset E' of $E(G)$, the subgraph of G whose vertex set is the set of vertices of the edges in E' and whose edge set is E', is called the subgraph of G induced by E' and denoted by $G[E']$. Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f: E(G) \rightarrow \mathrm{R}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges incident to vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$. Let b be a positive integer and let G have at least k vertices of degree at least b. A function $f: E(G) \rightarrow \{-1,1\}$ is called a *signed* (b,k)*-edge cover* (SbkEC) of G, if $f(v) \geq b$ for at least k vertices v of G. The *signed* (b,k) *-edge cover number* of a graph G is $\rho'_{b,k}(G) = \min \{\sum_{e \in E} f(e) \mid f \text{ is an SbkEC on } G\}$. The signed (b,k) -edge cover f of G with $f(E(G)) = \rho'_{b,k}(G)$ is called a $\rho'_{b,k}(G)$*-cover*. For any signed (b,k) -edge cover f of G we define

*Corresponding author

$P=\{e\in E \mid f(e)=1\}$, $M=\{e\in E \mid f(e)=-1\}$, $V^{+}=\{v\in V \mid f(v)\ge b\}$ and $V^{-}=\{v\in V \mid f(v)<b\}$.

If $b=1$ and $k=n$, then the signed (b,k)-edge cover number is called the *signed star domination number*. The signed star domination number was introduced by Xu in [7] and denoted by $\gamma_{SS}'(G)$. The signed star domination number has been studied by several authors (see for example [7,10]).

If $b=1$ and $1\le k\le n$, then the signed (b,k)-edge cover number is called the *signed star k-subdomination number*. The signed star k-subdomination number was introduced by Saei and Sheikholeslami in [11] and denoted by $\gamma_{SS}^{k}(G)$.

If b is an arbitrary positive integer and $k=n$, then the signed (b,k)-edge cover number is called the *signed b-edge cover number*. The signed b-edge cover number was introduced by Bonato *et al.* in [12] and denoted by $\rho_b'(G)$.

The purpose of this paper is to initiate the study of the signed (b,k)-edge cover number $\rho_{b,k}'(G)$. Here are some well-known results on $\gamma_{SS}'(G)$, $\gamma_{SS}^{k}(G)$ and $\rho_b'(G)$.

Theorem 1 [10] For every graph G of order $n\ge 4$, $\rho_{1,n}'(G)\le 2n-4$.

Theorem 2 [11] For every graph G of order $n\ge 4$ without isolated vertices, $\rho_{1,k}'(G)\le n+k-4.$

Theorem 3 [10] For every graph G of order n without isolated vertices, $\rho_{1,n}'(G)\ge\lceil\frac{n}{2}\rceil$.

Theorem 4 [11] For every graph G of order $n\ge 2$ without isolated vertices,

$$\rho_{1,k}'(G)\ge\left\lceil\frac{(\Delta(G)+1)k-n\Delta(G)}{2}\right\rceil.$$

Theorem 5 [12] Let b be a positive integer. For every graph G of order n and minimum degree at least b,

$$\rho_{b,n}'(G)\ge\left\lceil\frac{bn}{2}\right\rceil.$$

We make use of the following result in this paper.

Theorem 6 [7] Every graph G with $\delta(G)\ge 3$ contains an even cycle.

2. Lower Bounds for SbkECN of Graphs

In this section we present some lower bounds on $\rho_{b,k}'$ in terms of the order, the size, the maximum degree and the degree sequence of G. Our first proposition is a generalization of Theorems 3, 4 and 5.

Proposition 1 Let G be a graph of order n without isolated vertices and maximum degree $\Delta=\Delta(G)$. Let b be a positive integer and let $n_0\ge 1$ be the number of vertices with degree at least b. Then for every positive integer $1\le k\le n_0$,

$$\rho_{b,k}'(G)\ge\frac{k(b+\Delta)-n_0(\Delta-b+1)-n(b-1)}{2}.$$

Proof. Let f be a $\rho_{b,k}'(G)$-cover. We have

$$\begin{aligned}\rho_{b,k}'(G) &= \sum_{e\in E(G)} f(e)=\frac{1}{2}\sum_{v\in V(G)}\sum_{e\in E(v)} f(e)\\ &= \frac{1}{2}\sum_{v\in V^{+}}\sum_{e\in E(v)} f(e)+\frac{1}{2}\sum_{v\in V^{-}}\sum_{e\in E(v)} f(e)\\ &\ge \frac{kb}{2}-\frac{(n_0-k)\Delta+(n-n_0)(b-1)}{2}\\ &= \frac{k(b+\Delta)-n_0(\Delta-b+1)-n(b-1)}{2}.\end{aligned}$$

Theorem 2 Let G be a graph of order n, size m, without isolated vertices and with degree sequence $(d_1,d_2,\dots,d_n)$, where $d_1\le d_2\le\dots\le d_n$. Let b be a positive integer and let $n_0\ge 1$ be the number of vertices with degree at least b. Then for every positive integer $1\le k\le n_0$,

$$\rho_{b,k}'(G)\ge\frac{\sum_{j=1}^{k}(bd_j+d_j^2)}{2d_n}-m.$$

Proof. Let g be a $\rho_{b,k}'(G)$-cover of G and let $g(v)\ge b$ for k distinct vertices v in $B(G)=\{v_{j_1},\dots,v_{j_k}\}$. Define $f:E(G)\to\{0,1\}$ by $f(e)=(g(e)+1)/2$ for each $e\in E(G)$. We have

$$\sum_{e\in E(G)} f(N_G[e])=\sum_{e=uv\in E(G)}\frac{g(N_G[e])+\deg(u)+\deg(v)-1}{2}. \qquad (1)$$

Since

$$\sum_{e\in E(G)}(g(N_G[e])+g(e))=\sum_{v\in V}g(E(v))\deg(v)$$

and

$$\sum_{e=uv\in E(G)}(\deg(u)+\deg(v))=\sum_{v\in V}\deg(v)^2,$$

by (1) it follows that

$$\begin{aligned}
&\sum_{e\in E(G)} f(N_G[e])\\
&=\frac{1}{2}\sum_{v\in V}\deg(v)(g(E(v))+\deg(v))-\frac{1}{2}\sum_{e\in E(G)}g(e)-\frac{m}{2}\\
&\geq\frac{1}{2}\sum_{v\in V\setminus\{v_{j_1},\dots,v_{j_k}\}}\deg(v)(g(E(v))+\deg(v))+\\
&\quad\frac{1}{2}\sum_{i=1}^{k}(bd_{j_i}+d_{j_i}^2)-\frac{1}{2}\rho'_{b,k}(G)-\frac{m}{2}\\
&\geq\frac{1}{2}\sum_{i=1}^{k}(bd_{j_i}+d_{j_i}^2)-\frac{1}{2}\rho'_{b,k}(G)-\frac{m}{2}\\
&\geq\frac{1}{2}\sum_{j=1}^{k}(bd_{j}+d_{j}^2)-\frac{1}{2}\rho'_{b,k}(G)-\frac{m}{2}.
\end{aligned}\tag{2}$$

On the other hand,

$$\begin{aligned}
\sum_{e\in E(G)} f(N_G[e]) &= \sum_{v\in V}f(E(v))\deg(v)-\sum_{e\in E(G)}f(e)\\
&\leq \sum_{v\in V}f(E(v))d_n-\sum_{e\in E(G)}f(e)\\
&= d_n(2\sum_{e\in E(G)}f(e))-\sum_{e\in E(G)}f(e)\\
&= (2d_n-1)\sum_{e\in E(G)}f(e).
\end{aligned}\tag{3}$$

By (2) and (3)

$$\sum_{e\in E(G)}f(e)\geq\frac{\frac{1}{2}\sum_{j=1}^{k}(bd_j+d_j^2)-\frac{1}{2}\rho'_{b,k}(G)-\frac{m}{2}}{2d_n-1}.\tag{4}$$

Since $g(E(G))=2f(E(G))-m$, by (4)

$$\rho'_{b,k}(G)=\sum_{e\in E(G)}g(e)\geq$$

$$\frac{1}{2d_n-1}(\sum_{j=1}^{k}(bd_j+d_j^2)-\rho'_{b,k}(G)-m)-m.$$

Thus,

$$\rho'_{b,k}(G)\geq\frac{\sum_{j=1}^{k}(bd_j+d_j^2)}{2d_n}-m,$$

as desired.

An immediate consequence of Theorem 2 is:

Corollary 3 For every r-regular graph G of size m, $\rho'_{b,k}(G)\geq\frac{k(b+r)}{2}-m.$ Furthermore, the bound is sharp for r-regular graphs with $b=r$ and $k=n$.

Theorem 4 Let G be a graph of order $n\geq 2$, size m, without isolated vertices, with minimum degree $\delta=\delta(G)$ and maximum degree $\Delta=\Delta(G)$. Let b be a positive integer and $n_0\geq 1$ be the number of vertices with degree at least b. Then for each positive integer $1\leq k\leq n_0$

$$\rho_{b,k}{}'(G)\geq$$

$$\frac{(\Delta^2+b^2)k-2(\Delta-\delta)m-(b-1)^2n-(\Delta^2-(b-1)^2)n_0}{2\delta}.\tag{5}$$

Furthermore, the bound is sharp for n-cycles when $b=2$ and $k=n$.

Proof. Let $B(G)=\{v\in V(G)\,|\,\deg(v)\geq b\}$ and let f be a $\rho'_{b,k}(G)$-cover. Since for each $v\in V^+$, $f(v)\geq b$, it follows that $|M\cap E(v)|\leq\lfloor\frac{\deg(v)-b}{2}\rfloor$. Thus

$$\begin{aligned}
&(2\delta-1)|M|\\
&\leq\sum_{e=uv\in M}(\deg(u)+\deg(v)-1)\\
&=-|M|+\sum_{e=uv\in M}(\deg(u)+\deg(v))\\
&=-|M|+\sum_{v\in V(G[M])}|M\cap E(v)|\deg(v)\\
&\leq-|M|+\sum_{v\in V^+}|M\cap E(v)|\deg(v)+\sum_{v\in V^-}|M\cap E(v)|\deg(v)\\
&\leq-|M|+\sum_{v\in V^+}\lfloor\frac{\deg(v)-b}{2}\rfloor\deg(v)+\sum_{v\in V^-}\deg(v)^2\\
&\leq-|M|+\sum_{v\in V^+}\frac{\deg(v)^2}{2}+\sum_{v\in V^-}\deg(v)^2-\frac{b}{2}\sum_{v\in V^+}\deg(v)
\end{aligned}$$

$$\leq -|M| + \sum_{v\in V}\frac{\deg(v)^2}{2} + \sum_{v\in V^-}\frac{\deg(v)^2}{2} - \frac{b^2}{2}|V^+|$$

$$\leq -|M| + \Delta\sum_{v\in V}\frac{\deg(v)}{2} + \sum_{v\in V^-\cap B(G)}\frac{\deg(v)^2}{2} +$$

$$\sum_{v\in V^-\setminus B(G)}\frac{\deg(v)^2}{2} - \frac{b^2}{2}|V^+|$$

$$\leq -|M| + \Delta m - \frac{b^2}{2}k + \frac{\Delta^2}{2}|V^-\cap B(G)| + \frac{(b-1)^2}{2}|V^-\setminus B(G)|$$

$$\leq -|M| + \Delta m - \frac{b^2}{2}k + \frac{\Delta^2}{2}(n_0 - k) + \frac{(b-1)^2}{2}(n - n_0)$$

Hence,

$$|M| \leq \frac{\Delta m}{2\delta} + \frac{1}{4\delta}((b-1)^2 n + (\Delta^2 - (b-1)^2)n_0 - (\Delta^2 + b^2)k).$$

Now (5) follows by the fact that $\rho'_{b,k}(G) = m - 2|M|$.

3. An Upper Bound on SbkECN

Bonato *et al.* in [11] posed the following conjecture on $\rho'_b(G)$.

Conjecture 5 Let $b \geq 2$ be an integer. There is a positive integer n_b so that for any graph G of order $n \geq n_b$ with minimum degree b,

$$\rho'_b(G) \leq (b+1)(n-b-1).$$

Since $\rho'_b(K_{b+1,n-b-1}) = (b+1)(n-b-1)$, the upper bound would be the best possible if the conjecture were true. They also proved that the conjecture is true for $b = 2$. In this section we provide an upper bound for $\rho_{b,k}'(G)$, where $b = 2$ and $1 \leq k \leq n$. The proof of the next theorem is essentially similar to the proof of Theorem 5 in [11].

Theorem 6 *Let G be a graph of order n, size m and without isolated vertices. Let $n_0 > 0$ be the number of vertices with degree at least 2. Then for $n \geq 7$ and $1 \leq k \leq n_0$,*

$$\rho'_{2,k}(G) \leq 2n + k - 9.$$

Proof. The proof is by induction on the size *m* of *G*. By a tedious and so omitted argument, it follows that $\rho'_{2,k}(G) \leq k+5$ if $n = 7$. We may therefore assume that $n \geq 8$. Suppose that the theorem is true for all graphs G without isolated vertices and size less than m. Let G be a graph of order $n \geq 8$, size m and without isolated vertices. We will prove that $\rho'_{2,k}(G) \leq 2n + k - 9$ for each $1 \leq k \leq n_0$. We consider four cases.

Case 1. $\delta(G) = 1$.

Let u be a vertex of degree 1 and $v \in N(u)$. First suppose $\deg(v) = 1$. Then the induced subgraph $G[u,v]$ is K_2. It is straightforward to verify that $\rho'_{2,k} \leq 2n + k - 9$ when $n = 8$. Hence, we may assume that $n \geq 9$. Let $G' = G - uv$. Then G' is a graph of order $n - 2 \geq 7$, size $m - 1$ and without isolated vertices. By the inductive hypothesis, $\rho'_{2,k}(G') \leq 2(n-2) + k - 9 = 2n + k - 13$. Let f be a $\rho'_{2,k}$ (G') -cover. Define $g: E(G) \to \{-1,1\}$ by $g(uv) = -1$ and $g(e) = f(e)$ if $e \in E(G) - uv$. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G) \leq \rho'_{2,k}(G') - 1 \leq 2n + k - 14 < 2n + k - 9.$$

Now suppose $\deg(v) \geq 2$. Consider two subcases.

Subcase 1.1 $\deg(v) \geq 3$.

By the inductive hypothesis on $G - u$, $\rho'_{2,k}(G-u) \leq 2(n-1) + k - 9 = 2n + k - 11$. Let f be a $\rho'_{2,k}$ $(G-u)$ -cover and define $g: E(G) \to \{-1,1\}$ by $g(uv) = 1$ and $g(e) = f(e)$ if $e \in E(G) - uv$. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G) \leq \rho'_{2,k}(G') + 1 \leq 2n + k - 10 < 2n + k - 9.$$

Subcase 1.2 $\deg(v) = 2$.

Let $w \in N(v) - \{u\}$. If $k = 1$, then define $g: E(G) \to \{-1,1\}$ by $g(uv) = g(vw) = 1$ and $g(e) = -1$ if $e \in E(G) \setminus \{uv, vw\}$. Obviously, g is a S2kEC of G and we have

$$\rho'_{2,k}(G) \leq g(E(G)) = 4 - m \leq 2n + k - 9.$$

Let $k \geq 2$. It follows that $n_0 \geq 2$. By the inductive hypothesis on $G - \{u\}$, $\rho'_{2,k-1}(G - \{u\}) \leq 2(n-1) + (k-1) - 9 = 2n + k - 12$. Let f be a $\rho'_{2,k-1}$ $(G - \{u\})$ -cover. Define $g: E(G) \to \{-1,1\}$ by $g(uv) = g(vw) = 1$ and $g(e) = f(e)$ if $e \in E(G) \setminus \{uv, vw\}$. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G) \leq \rho'_{2,k-1}(G - \{u\}) + 3 \leq 2n + k - 9.$$

Case 2. $\delta(G)=2$.

Let w be a vertex of degree 2 and $N(w)=\{u,v\}$. Consider two subcases.

Subcase 2.1 $uv\notin E(G)$. Let G' be the graph obtained from $G-\{w\}$ by adding an edge uv. Then G' has order $n-1$, size $m-1$ and at least $k-1$ vertices with degree at least 2. By the inductive hypothesis,

$$\rho'_{(k-1),2}(G')\le 2(n-1)+(k-1)-9=2n+k-12.$$

Let f be a $\rho'_{2,k}(G')$-cover. Define $g:E(G)\to\{-1,1\}$ by $g(uw)=g(vw)=1$ and $g(e)=f(e)$ if $e\in E(G)\setminus\{uv,vw\}$. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G)\le g(E(G))\le f(E(G'))+3\le 2n+k-9.$$

Subcase 2.2 $uv\in E(G)$. First let both u and v have degree 2. Then the induced subgraph $G[\{u,v,w\}]$ is an isolated triangle. If $1\le k\le 3$, then define $f:E(G)\to\{-1,1\}$ by

$$f(uv)=f(vw)=f(uw)=1 \text{ and } f(e)=-1 \text{ otherwise.}$$

Then

$$\rho'_{2,k}(G)\le f(E(G))=6-m\le 2n+k-9.$$

Now suppose that $k\ge 4$. It is not hard to show that $\rho'_{2,k}(G)\le 2n+k-9$ when $n=8$ or 9. Hence, we may assume that $n\ge 10$. Let $G'=G\setminus\{u,v,w\}$. Then G' is a graph of order $n-3\ge 7$, size $m-3$ and has at least $k-3$ vertices with degree at least 2. By the inductive hypothesis, $\rho'_{2,(k-3)}(G')\le 2(n-3)+(k-3)-9=2n+k-18$. Let f be a $\rho'_{2,(k-3)}(G')$-cover. Define $g:E(G)\to\{-1,1\}$ by

$$g(uv)=g(vw)=g(uw)=1 \text{ and } g(e)=f(e) \text{ if } e\in E(G').$$

Obviously, g is a S2kEC of G and

$$\rho'_{2,k}(G)=g(E(G))\le f(E(G'))+3\le(2n+k-18)+3.$$

Now let $\min\{\deg(u),\deg(v)\}\ge 3$. If $k=1$, define $g:E(G)\to\{-1,1\}$ by $g(uw)=g(vw)=1$ and $g(e)=-1$ otherwise. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G)\le g(E(G))=4-m<2n+k-9.$$

If $k\ge 2$, then $G'=G-\{w\}$ is a graph of order $n-1$, size $m-2$ and has at least $k-1$ vertices with degree at least 2. By the inductive hypothesis, we have that $\rho'_{2,(k-1)}(G')\le 2n+k-12$. Let f be a $\rho'_{2,(k-1)}(G')$-cover. We can obtain a S2kEC g of G by assigning $g(e)=1$ for each $e\in E(G)\setminus E(G')$ and $g(e)=f(e)$ for each $e\in E(G')$. Then we have

$$g(E(G))=f(E(G'))+2=\rho'_{2,(k-1)}(G')+2<2n+k-9.$$

Hence, $\rho'_{2,k}(G)<2n+k-9$, as desired.

Finally, assume $\min\{\deg(u),\deg(v)\}=2$. Let without loss of generality $\deg(u)=2$. If $1\le k\le 2$, define $g:E(G)\to\{-1,1\}$ by $g(uw)=g(vw)=g(uv)=1$ and $g(e)=-1$ otherwise. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G)\le g(E(G))=6-m\le 2n+k-9.$$

If $k\ge 3$, then $G'=G-\{w\}$ is a graph of order $n-1$, size $m-2$ and has at least $k-2$ vertices with degree at least 2. By the inductive hypothesis,

$$\rho'_{2,(k-2)}(G')\le 2(n-1)+(k-2)-9=2n+k-13.$$

Let f be a $\rho'_{2,(k-2)}(G')$-cover. Define $g:E(G)\to\{-1,1\}$ by

$$g(uv)=g(vw)=g(uw)=1 \text{ and } g(e)=f(e)$$
$$\text{if } e\in E(G')\setminus\{uv\}.$$

Obviously, g is a S2kEC of G and

$$\rho'_{2,k}(G)\le g(E(G))=f(E(G'))+4\le 2n+k-9.$$

Case 3. $\delta(G)=3$.

Let w be a vertex with degree 3. If $k=1$, define $g:E(G)\to\{-1,1\}$ by $g(uw)=1$ if $u\in N(w)$ and $g(e)=-1$ otherwise. Obviously, g is a S2kEC and so

$$\rho'_{2,k}(G)\le g(E(G))=6-m<2n+k-9.$$

If $k\ge 2$, then $G'=G-\{w\}$ is a graph of order $n-1$, size $m-3$ and has at least $k-1$ vertices with degree at least 2. By the inductive hypothesis, we have that $\rho'_{2,(k-1)}(G')\le 2n+k-12$. Let f be a $\rho'_{2,(k-1)}(G')$-cover. We can obtain a S2kEC g of G by assigning $g(e)=1$ for each $e\in E(G)\setminus E(G')$ and $g(e)=f(e)$ for each $e\in E(G')$. Then we have

$$g(E(G))=f(E(G'))+3=\rho'_{2,(k-1)}(G')+3\le 2n+k-9.$$

Hence, $\rho'_{2,k}(G)\le 2n+k-9$, as desired.

Case 4. $\delta(G)\ge 4$.

Then G has an even cycle by Theorem 6. Let $C=(v_1,v_2,\ldots,v_s)$ be an even cycle in G. Obviously, $G'=G-E(C)$ is a graph of order n, size $m-|E(C)|$ and has at least k vertices with degree at least 2. By the inductive hypothesis, $\rho'_{2,k}(G')\le 2n+k-9$. Let f be a $\rho'_{2,k}(G')$-cover. Let $v_{s+1}=v_1$ and define $g:E(G)\to\{-1,1\}$ by

$$g(v_iv_{i+1})=(-1)^i \text{ if } i=1,\ldots,s \text{ and } g(e)=f(e) \text{ for } e\in E(G)\setminus E(C).$$

Obviously, g is a S2kEC and hence $\rho'_{2,k}(G)=\rho'_{2,k}(G')\le 2n+k-9$. This completes the proof.

4. Conclusions

In this paper we initiated the study of the signed (b,k)-edge cover numbers for graphs, generalizing the signed star domination numbers, the signed star *k*-domination numbers and the signed *b*-edge cover numbers in graphs. The first lower bound obtained in this paper for the signed (b,k)-edge cover number concludes the existing lower bounds for the other three parameters. Our upper bound for the signed (b,k)-edge cover number also implies the existing upper bound for the signed *b*-edge cover number. Finally, Theorem 6 inspires us to generalize Conjecture 5.

Conjecture 7 Let $b\ge 3$ be an integer. There is a positive integer n_b so that for any graph *G* of order $n\ge n_b$ with $n_0\ge 1$ vertices of degree at least *b*, and for any integer $1\le k\le n_0$, $\rho'_{b,k}(G)\le bn+k-(b+1)^2$.

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