Computing the metric dimension for chain graphs*

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Abstract. The metric dimension of a graph G is the smallest size of a set R of vertices that can distinguish each vertex pair of G by the shortest-path distance to some vertex in R. Computing the metric dimension is NP-hard, even when restricting inputs to bipartite graphs. We present a linear-time algorithm for computing the metric dimension for chain graphs, which are bipartite graphs whose vertices can be ordered by neighborhood inclusion.

1 Introduction

Let G be a graph, and let $\mathrm{dist}_G(\cdot,\cdot)$ denote the shortest-path distance of G. A *resolving set* for G is a set $R \subseteq V(G)$ so that each vertex pair u, v of G has a vertex $x \in R$ satisfying $\mathrm{dist}_G(x,u) \neq \mathrm{dist}_G(x,v)$. The metric dimension problem asks for a resolving set of smallest size, and the *metric dimension* of a graph is the size of such a smallest set. Metric dimension was independently defined by Harary and Melter [16] and by Slater [22]. Metric dimension and resolving sets have applications in several technical disciplines, such as chemistry [6], robot navigation [20], network discovery and verification [2], and strategies for the Mastermind game [8]. Metric dimension was also studied for graphs of a high degree of symmetry [1,4,12].

The Metric Dimension decision problem is among the classical NP-complete problems [14]. Already Harary and Melter as well as Slater showed that the metric dimension can be computed efficiently for trees [16,22]. Díaz, Pottonen, Serna, and van Leeuwen stated that Metric Dimension is hard on bounded-degree planar graphs [9]. Epstein, Levin, and Woeginger extended the hardness of Metric Dimension to split graphs, bipartite graphs, co-bipartite graphs, and line graphs of bipartite graphs, and they showed that the metric dimension can be computed efficiently for cycles, cographs, k-edge-augmented trees, and wheels [11]. They even extended their positive results to vertex-weighted graphs, asking for a resolving set of smallest weight instead of smallest size [11]. The metric dimension can be computed in polynomial time for outer-planar graphs [9]. Approximability and inapproximability results were given by Hauptmann, Schmied, and Viehmann [18]. Further hardness results are due to Hartung and Nichterlein [17]. Finally, formulas for metric dimension are known for special graphs like trees and hypercubes [19,21].

We study the metric dimension problem on a class of bipartite graphs. We consider bipartite graphs whose vertices admit an ordering by neighborhood-inclusion. Such graphs are called chain graphs. We show that the metric dimension can be computed in linear time for chain graphs, by evaluating an easy-to-compute function. We also give an exact formula for the metric dimension of the chain graphs without false twins, that is dependent on the number of vertices only.

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2 Graph preliminaries, resolving sets, and metric dimension

Our graphs are simple, finite, undirected. Let $G = (V, E)$ be a graph. The set of neighbors of a vertex v of G is denoted by $N_G(v)$. For a set $S \subseteq V$, $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$. We refer the reader to the monograph by Diestel [10] for standard graph terminology and notation that we do not define here.

The *distance* of two vertices u and v of G is denoted by $\text{dist}_G(u, v)$, and it is the length of a shortest u, v-path of G. If G has no u, v-path, then we let $\text{dist}_G(u, v) = |V(G)|$. For a vertex triple u, v, x of G, we say that x *resolves* u and v, or x is a *resolving vertex* for u and v, if $u = v$ or $\text{dist}_G(x, u) \neq \text{dist}_G(x, v)$. We will mostly implicitly assume $u \neq v$ when discussing resolving vertices. A *resolving set* for G is a set R of vertices of G that has a resolving vertex for each vertex pair of G. A *minimum resolving set* for G is a resolving set for G of smallest cardinality. The *metric dimension* of G, denoted as $\dim(G)$, is the size of a minimum resolving set for G.

For every $x \in V$, the *distance relation of* x, denoted as $\sim_x$, is the binary relation on V defined as follows: for every vertex pair u, v of G, $u \sim_x v$ if and only if $\text{dist}_G(x, u) = \text{dist}_G(x, v)$. Note that $\sim_x$ is an equivalence relation on V. Also note that x resolves u and v if and only if u and v belong to different equivalence classes of $\sim_x$, which means that u and v appear in different levels of a breadth-first search of G with source vertex x. By $\mathfrak{Eq}(x)$, we denote the set of the equivalence classes of $\sim_x$. For $X \subseteq V$ where X is non-empty, $\sim_X$ denotes the intersection of the relations $\sim_x$ for $x \in X$, i.e., $\sim_X = \bigcap_{x \in X} \sim_x$, and $\mathfrak{Eq}(X)$ denotes the set of the equivalence classes of $\sim_X$.

A *refinement* of a partition $\mathfrak{C}$ of V is a partition $\mathfrak{D}$ of V such that every member of $\mathfrak{D}$ is contained as a subset in a member of $\mathfrak{C}$. We write $\mathfrak{D} \sqsubseteq \mathfrak{C}$, if $\mathfrak{D}$ is a refinement of $\mathfrak{C}$, or $\mathfrak{D}$ *refines* $\mathfrak{C}$. Observe the two monotonicity properties: $\mathfrak{Eq}(V) \sqsubseteq \mathfrak{Eq}(X) \sqsubseteq \mathfrak{Eq}(Y)$ for $\emptyset \subset Y \subseteq X \subseteq V$, and for $X, Y \subseteq V$ and $Z \subseteq V$, if $\mathfrak{Eq}(X) \sqsubseteq \mathfrak{Eq}(Y)$ then $\mathfrak{Eq}(X \cup Z) \sqsubseteq \mathfrak{Eq}(Y \cup Z)$.

The following lemma shows a straightforward connection between resolving sets and the equivalence classes of the distance relation.

Lemma 1. *Let $G = (V, E)$ be a graph. A set $R \subseteq V$ is a resolving set for G if and only if the members of $\mathfrak{Eq}(R)$ are of size 1 each, which means $\mathfrak{Eq}(V) = \mathfrak{Eq}(R)$.*

Let $G = (V, E)$ be a graph, and let u and v be vertices of G. We call u and v *false twins* of G, or just *twins*, if $N_G(u) = N_G(v)$. Note that false twins are non-adjacent. The false-twin relation is an equivalence relation on V, called the *twin relation*, its equivalence classes are called *twin classes*, and the twin classes can be computed in linear time (see, for instance, [15]).

Observation 1 *Let $G = (V, E)$ be a graph, and let A be a twin class of G. For every $R \subseteq V$, $A \setminus R$ is contained as a subset in a member of $\mathfrak{Eq}(R)$, which means that R cannot resolve the twins in $A \setminus R$. Thus, if R is a resolving set for G, then $|A \setminus R| \leq 1$.*

An *isolated vertex* of G is a vertex without neighbors. The metric dimension of a disconnected graph can be computed from the metric dimension of its connected components, where isolated vertices play a special role.

Proposition 1 ([11]). *Let G be a graph, let $C_1, \ldots, C_t$ be the connected components of G, and let I be the set of isolated vertices of G. Then,*

$$\dim(G) \;=\; \dim(C_1) + \cdots + \dim(C_t) + \begin{cases} 0 & , \text{ if } I = \emptyset \\ |I| - 1 & , \text{ if } I \neq \emptyset. \end{cases}$$

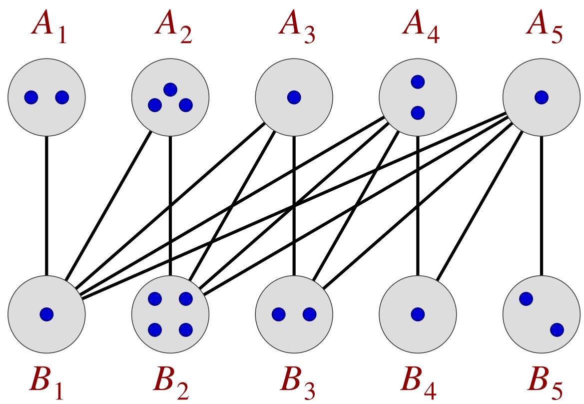

Fig. 1. The figure represents a connected chain graph. The two independent sets A and B are already partitioned into their twin classes, yielding $A_1, \ldots, A_5$ for A and $B_1, \ldots, B_5$ for B. The vertices in a twin class have the same neighbors. As an example, the vertices in B_3 are adjacent to the vertices in $A_3 \cup A_4 \cup A_5$. The associated cardinality sequences are $\langle 2, 3, 1, 2, 1\rangle$ for A and $\langle 1, 4, 2, 1, 2\rangle$ for B.

In view of Proposition 1, we want to point out that the metric dimension of a graph on a single vertex is 0. Due to Proposition 1, we can restrict ourselves to connected graphs. Furthermore, the metric dimension of a graph on two vertices is 1. *We henceforth assume that every graph in this paper is connected and has at least three vertices.*

In this paper, we consider a class of bipartite graphs. For convenience, we denote a bipartite graph $G = (V, E)$ as (A, B, E) where $\{A, B\}$ defines a partition of V into two independent sets. A *connected chain graph* on at least three vertices is a bipartite graph $G = (A, B, E)$ with its twin classes $A_1, \ldots, A_k, B_1, \ldots, B_l$ where $A_1 \cup \cdots \cup A_k = A$ and $B_1 \cup \cdots \cup B_l = B$ such that $\emptyset \subset N_G(A_1) \subset \cdots \subset N_G(A_k) = B$ and $A = N_G(B_1) \supset \cdots \supset N_G(B_l) \supset \emptyset$. It directly follows that $k = l$ must hold. An example of a connected chain graph with $2 \cdot 5$ twin classes is depicted in Figure 1.

We refer to the monograph by Brandstädt, Le, and Spinrad [3] for the standard definition and a more comprehensive treatment of chain graphs.

3 A special minimum resolving set for chain graphs

Let $G = (A, B, E)$ be a connected chain graph with its twin classes $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ that satisfy $N_G(A_1) \subset \cdots \subset N_G(A_k)$ and $N_G(B_1) \supset \cdots \supset N_G(B_k)$. We combine consecutive twin classes. For $1 \leq i \leq j \leq k$, we denote the union $A_i \cup \cdots \cup A_j$ shortly as $A_{i,\ldots,j}$, and $B_{i,\ldots,j} = B_i \cup \cdots \cup B_j$ analogously. We also choose a "representative" of each twin class: choose $a_i \in A_i$ and $b_i \in B_i$ for every $1 \leq i \leq k$.

As our main combinatorial result, we show that G has a minimum resolving set of a restricted structure. We prove this result by exploiting the structure of chain graphs. The following observations will be important: $N_G(a_p) = B_{1,\ldots,p}$ and $N_G(b_p) = A_{p,\ldots,k}$ for every $1 \leq p \leq k$, and $\mathrm{dist}_G(a, a') \in \{0, 2\}$ and $\mathrm{dist}_G(b, b') \in \{0, 2\}$ and $\mathrm{dist}_G(a, b) \in \{1, 3\}$ for $a, a' \in A$ and $b, b' \in B$. Thus, $|\mathfrak{Eq}(x)| \leq 4$ for every $x \in A \cup B$.

Lemma 2. *Let R be a resolving set for G, and let $1 \leq p \leq k$. Assume $B_p \subseteq R$ and $B_{1,\ldots,p-1} \not\subseteq R$. Then, there is an index j with $1 \leq j \leq k$ such that $(R \setminus \{b_p\}) \cup A_j$ is a resolving set for G.*

Proof. Observe

$$\mathfrak{Eq}(b_p) \;=\; \mathfrak{Eq}\big(\{b_p\}\big) = \big\{B \setminus \{b_p\},\ \{b_p\}\big\} \;\cup\; \big\{A_{1,\dots,p-1},\ A_{p,\dots,k}\big\}\,.$$

Recall that for $X \subseteq A \cup B$, the vertices in X are in singleton equivalence classes in $\mathfrak{Eq}(X)$. To simplify representations in the proof, we use rest-$\mathfrak{Eq}(X)$ to denote the equivalence classes in $\mathfrak{Eq}(X)$ that do not contain vertices from X. As an example, observe

$$\text{rest-}\mathfrak{Eq}\big(\{b_p\}\big) = \big\{B \setminus \{b_p\},\ A_{1,\dots,p-1},\ A_{p,\dots,k}\big\}\,.$$

For the proof, we determine an index j and a subset F of $(R \setminus \{b_p\}) \cup A_j$ such that $\mathfrak{Eq}(F) \sqsubseteq \mathfrak{Eq}(b_p)$. Then, $\mathfrak{Eq}((R\setminus\{b_p\})\cup A_j) \sqsubseteq \mathfrak{Eq}(R)$ directly follows from the monotonicity properties, and $(R \setminus \{b_p\}) \cup A_j$ is a resolving set for G due to Lemma 1.

We distinguish between two easy and two harder cases. Let $B'_p = B_p \setminus \{b_p\}$, and if $p \geq 2$, let $A'_{p-1} = A_{p-1} \cap R$ and $A''_{p-1} = A_{p-1} \setminus R$. If $p = 1$, then

$$\text{rest-}\mathfrak{Eq}\big(B'_1 \cup \{a_1\}\big) \;=\; \big\{B \setminus B_1, \{b_1\}\big\} \;\cup\; \big\{A \setminus \{a_1\}\big\}\,,$$

and if $p \geq 2$ and $A'_{p-1} \neq \emptyset$, then

$$\text{rest-}\mathfrak{Eq}\big(B'_p \cup A'_{p-1} \cup A_{p,\dots,k}\big) \;\sqsubseteq\; \big\{B_{1,\dots,p-1},\ \{b_p\},\ B_{p+1,\dots,k}\big\} \;\cup\; \big\{A_{1,\dots,p-2} \cup A''_{p-1}\big\}\,.$$

So, if $p = 1$, then $\mathfrak{Eq}\big(B'_1 \cup \{a_1\}\big) \sqsubseteq \mathfrak{Eq}(b_1)$, and $(R \setminus \{b_1\}) \cup A_1$ is a resolving set for G, and if $p \geq 2$ and $A'_{p-1} \neq \emptyset$ and $A_{p,\dots,k} \subseteq R \cup A_j$ for some $p \leq j \leq k$, then $(R \setminus \{b_p\}) \cup A_j$ is a resolving set for G. This completes the two easy cases.

We assume that no easy case applies, which means $p \geq 2$, and $A'_{p-1} = \emptyset$ or $A_{p,\dots,k} \not\subseteq R \cup A_j$ for every $p \leq j \leq k$. Our two harder cases distinguish between $A'_{p-1} \neq \emptyset$ and $A'_{p-1} = \emptyset$.

As the first harder case, assume $A'_{p-1} \neq \emptyset$. Let q with $p \leq q \leq k$ be smallest such that $A_q \not\subseteq R$, and let r with $q < r \leq k$ be smallest such that $A_r \not\subseteq R$; observe that q and r indeed exist. Let $u \in A_q \setminus R$ and $v \in A_r \setminus R$. Since R is a resolving set for G, there is a vertex $z \in R$ resolving u and v. Observe $z \in B_{q+1,\dots,r}$. Then,

$$\begin{aligned}
&\text{rest-}\mathfrak{Eq}\big(B'_p \cup A'_{p-1} \cup A_{p,\dots,r-1} \cup \{z\}\big)\\
&\sqsubseteq\; \big\{B_{1,\dots,p-1},\ \{b_p\},\ (B_{p+1,\dots,k}) \setminus \{z\}\big\} \;\cup\; \big\{A_{1,\dots,p-2} \cup A''_{p-1},\ A_{r,\dots,k}\big\}\,,
\end{aligned}$$

and $\mathfrak{Eq}\big(B'_p \cup A'_{p-1} \cup A_{p,\dots,r-1} \cup \{z\}\big) \sqsubseteq \mathfrak{Eq}(b_p)$, and $(R \setminus \{b_p\}) \cup A_q$ is a resolving set for G.

As the second harder case, assume $A'_{p-1} = \emptyset$. This means $A''_{p-1} = A_{p-1}$ by the definition of A''_{p-1} and $A_{p-1} = \{a_{p-1}\}$ due to Observation 1. We consider the vertices in A and in B separately. We begin with the vertices in A. Let s be smallest with $1 \leq s \leq p-1$ such that $A_{s,\dots,p-1} \subseteq R \cup A_{p-1}$. If $s = 1$, then

$$\text{rest-}\mathfrak{Eq}\big(A_{1,\dots,p-1}\big) \;\sqsubseteq\; \big\{B_{1,\dots,p-1},\ B_{p,\dots,k}\big\} \;\cup\; \big\{A_{p,\dots,k}\big\}\,.$$

Assume $s \geq 2$. Let $u \in A_{s-1} \setminus R$. Since R is a resolving set for G, there is a resolving vertex z in R for u and a_{p-1}. Observe $z \in B_{s,\dots,p-1}$. Then,

$$\text{rest-}\mathfrak{Eq}\big(A_{s,\dots,p-1} \cup \{z\}\big) \;\sqsubseteq\; \big\{(B_{1,\dots,p-1}) \setminus \{z\},\ B_{p,\dots,k}\big\} \;\cup\; \big\{A_{1,\dots,s-1},\ A_{p,\dots,k}\big\}\,.$$

Observe that each of the two situations refines $\{B_{1,\dots,p-1}, B_{p,\dots,k}, A_{1,\dots,p-1}, A_{p,\dots,k}\}$ already, and it remains to extract $\{b_p\}$ from $B_{p,\dots,k}$. We consider the vertices in B. If $A_p \cap R \neq \emptyset$, then

$$\text{rest-}\mathfrak{Eq}\big(B'_p \cup (A_p \cap R)\big) \;\sqsubseteq\; \big\{B_{1,\dots,p-1} \cup \{b_p\},\ B_{p+1,\dots,k}\big\} \;\cup\; \big\{A \setminus (A_p \cap R)\big\}\,,$$

and if $B_{p+1,\dots,k} \subseteq R$, then

$$\text{rest-}\mathfrak{Eq}\big(B'_p \cup B_{p+1,\dots,k}\big) \;\sqsubseteq\; \big\{B_{1,\dots,p-1} \cup \{b_p\},\; A\big\}\,.$$

Otherwise, assume $A_p \cap R = \emptyset$ and $B_{p+1,\dots,k} \not\subseteq R$. Let q with $q > p$ be smallest such that $B_q \not\subseteq R$. Let $v \in B_{p-1} \setminus R$ and $w \in B_q \setminus R$. Recall from the assumptions about p in the lemma that v does indeed exist. Since R is a resolving set for G, there is a resolving vertex z' in R for v and w, and $z' \in A_{p-1,\dots,q-1}$. However, since $(A_{p-1} \cup A_p) \cap R = \emptyset$, $z' \in A_{p+1,\dots,q-1}$ follows. So,

$$\text{rest-}\mathfrak{Eq}\big(B'_p \cup B_{p+1,\dots,q-1} \cup \{z'\}\big) \;\sqsubseteq\; \big\{B_{1,\dots,p-1} \cup \{b_p\},\; B_{q,\dots,k},\; A \setminus \{z'\}\big\}\,,$$

and we conclude that $(R \setminus \{b_p\}) \cup A_{p-1}$ is a resolving set for G. □

From Lemma 2, we obtain our main combinatorial result about a minimum resolving set of special structure.

Proposition 2. *G has a minimum resolving set R such that $B_i \not\subseteq R$ for every $1 \le i \le k$.*

Proof. Let R be a minimum resolving set for G, and assume that R is chosen such that $p = \min\{i : B_i \subseteq R\} \cup \{k+1\}$ is largest possible. If $p = k+1$, then R satisfies the condition of the claim.

Otherwise, $1 \le p \le k$, and we can apply Lemma 2: there is an index j with $1 \le j \le k$ such that $(R \setminus \{b_p\}) \cup A_j$ is a resolving set for G. Since $|A_j \setminus R| \le 1$ due to Observation 1, we observe $\big|(R \setminus \{b_p\}) \cup A_j\big| \le \big|R\big|$, and thus, $(R \setminus \{b_p\}) \cup A_j$ is a minimum resolving set for G, that contradicts the choice of R. □

4 Computing a minimum resolving set for chain graphs

We want to efficiently compute minimum resolving sets for chain graphs, and thus determine their metric dimension. We could devise a dynamic-programming algorithm based on our main combinatorial result, which is Proposition 2. However, we can do better. We explicitly define a minimum resolving set that also satisfies the structural property of Proposition 2 and explain how it can be computed in linear time.

Before we procede, note the following: for $G = (A, B, E)$ a chain graph, the number of edges of G is of order $\frac{|A| \cdot |B|}{2}$, and thus, linear running time for G is equivalent to a running time of order $(|A| + |B|)^2$. For algorithmic purposes, a more succinct representation is often chosen. In case of chain graphs, the structure of a connected chain graph is completely determined by the two cardinality sequences of the twin classes, and our metric dimension algorithm will have linear running time even on such a succinct input.

We consider a connected chain graph $G = (A, B, E)$ on at least three vertices and with its twin classes $A_1, \dots, A_k$ and $B_1, \dots, B_k$ where $N_G(A_1) \subset \cdots \subset N_G(A_k)$ and $N_G(B_1) \supset \cdots \supset N_G(B_k)$. The associated cardinality sequences are $\big\langle |A_1|, \dots, |A_k| \big\rangle$ and $\big\langle |B_1|, \dots, |B_k| \big\rangle$.

Definition 1. *Let $1 \le p \le q \le k$. We call $[p, q]$ a* segment *of G if $|B_p| = \cdots = |B_q| = 1$, and $p > 1$ implies $|B_{p-1}| \ge 2$, and $q < k$ implies $|B_{q+1}| \ge 2$.*

We can say that a segment is a maximal interval of consecutive twin classes in B each of size 1. Observe that size-1 twin classes in B play a special role for our minimum resolving sets according to Proposition 2, since a corresponding resolving set would not contain any vertex from such a twin class. Segments force properties of resolving sets, as we show in the next lemma.

Lemma 3. *Let $[p,q]$ be a segment of G, and let s,t be such that $p-1 \le s < t \le q$ and $s \ge 1$. Let $Q \subseteq A \cup B$, and assume that $Q \cap (B_p \cup \cdots \cup B_q) = \emptyset$. If Q is a resolving set for G, then $A_s \subseteq Q$ or $A_t \subseteq Q$.*

Proof. Assume $A_s \not\subseteq Q$ and $A_t \not\subseteq Q$. Let $u \in A_s \setminus Q$ and $v \in A_t \setminus Q$. Every resolving vertex for u and v is from $\{u,v\} \cup B_{s+1} \cup \cdots \cup B_t$. Since $Q \cap (\{u,v\} \cup B_{s+1} \cup \cdots \cup B_t) = \emptyset$, Q is not a resolving set for G. □

Definition 2. *Let $1 \le j \le k$, and let $[p,q]$ be a segment of G.*

1) *We call j a* high position *if $|A_j| \ge 2$ and $|B_j| \ge 2$ and if $j < k$ then $|B_{j+1}| \ge 2$.*
2) *We call j a* heavy position *in $[p,q]$ if $p-1 \le j \le q$ and $|A_j| \ge 2$.*
3) *We call $[p,q]$* good *if there is a heavy position in $[p,q]$.*

We can say that a high position is like a heavy position that is not in the "range" of a segment.

We now prove a lower bound on the metric dimension of G. To state this bound, we need some more notations.

- Let g be the number of good segments of G, and
  let h be the number of high positions.
- Let $K = A_k$ if $|A_k| = 1$ and $[p,k]$ is not a good segment of G for every $1 \le p \le k$, and let $K = \emptyset$ otherwise.

We clarify these notions for the chain graph of Figure 1: $[1,1]$ and $[4,4]$ are segments, and 1 is a heavy position in $[1,1]$ and 4 is a heavy position in $[4,4]$, and $K = A_5$. Moreover, 2 is a high position, since $|A_2|, |B_2|, |B_3| \ge 2$, and it is the only high position.

Lemma 4. *The metric dimension of G is at least $|A \cup B| - k - g - h - |K|$.*

Proof. Let R^* be a minimum resolving set for G that satisfies the structural property of Proposition 2. Hence, $|R^* \cap B| = |B| - k$, and $|A_j \setminus R^*| \le 1$ for every $1 \le j \le k$ due to Observation 1, so that $|A| - k \le |R^* \cap A| \le |A|$.

Let $L = \{j : 1 \le j \le k \text{ and } A_j \not\subseteq R^*\}$, and observe

$$|L| = |A \setminus R^*| = |A \setminus (R^* \cap A)| = |A| - |R^* \cap A| \,.$$

If $|L| \le g + h + |K|$, then

$$\begin{aligned} \dim(G) \;&=\; |R^*| = |R^* \cap A| + |R^* \cap B| \\ &= |A| - |L| + |B| - k \;\ge\; |A| + |B| - k - g - h - |K| \,, \end{aligned}$$

which is the desired lower bound. We verify $|L| \le g + h + |K|$.

Claim. $|A_j| \ge 2$ for every $j \in L \setminus \{k\}$.

Proof. Let $j \in L \setminus \{k\}$, and consider $u \in B_j \setminus R^*$ and $v \in B_{j+1} \setminus R^*$, that exist according to the assumptions about R^*. If $A_j \cap R^* = \emptyset$, then R^* contains no resolving vertex for u and v, so that $A_j \cap R^* \ne \emptyset$ follows, and $A_j \not\subseteq R^*$ implies $|A_j| \ge 2$. ◊

If $|K| = 0$, then let $L' = L$, and if $|K| = 1$, then let $L' = L \setminus \{k\}$. Observe that $|L| = |L'| + |K|$.

We show $|L'| \le g + h$, which implies $|L| = |L'| + |K| \le g + h + |K|$. Let $r \in L'$, and assume that r is not a high position. If $r < k$, then $|A_r| \ge 2$ by the Claim, and $|B_r| = 1$ or $|B_{r+1}| = 1$, and if $r = k$, then $|B_k| = 1$. It follows: G has a segment $[p,q]$ such that $p-1 \le r \le q$. Observe:

$[p, q]$ is a good segment of G, because of $|A_r| \geq 2$ in case when $r < k$, and because of $|K| = 0$ in case when $r = k$.

We show $L' \cap \{p-1, \ldots, q\} = \{r\}$. If $|L' \cap \{p-1, \ldots, q\}| \geq 2$, then there are $s, t \in L'$ such that $p-1 \leq s < t \leq q$ and $A_s \not\subseteq R^*$ and $A_t \not\subseteq R^*$, and R^* is not a resolving set for G due to Lemma 3, a contradiction. So, $|L' \cap \{p-1, \ldots, q\}| = 1$, and $r \in L' \cap \{p-1, \ldots, q\}$ implies the result. □

We now show the complementary result of Lemma 4, that is an upper bound on the metric dimension of G. We prove the upper bound by explicitly describing a minimum resolving set for G. Arbitrarily choose representatives $a_i \in A_i$ and $b_i \in B_i$ for every $1 \leq i \leq k$. We define the following sets of vertices of G:

$$H = \{a_j : 1 \leq j \leq k \text{ and } j \text{ is a high position}\}$$
$$S = \{a_j : j \text{ is a selected heavy position in a good segment}\}$$
$$R = (A \cup B) \setminus \big(\{b_1, \ldots, b_k\} \cup H \cup S \cup K\big)\,.$$

Recall that each good segment contains a heavy position. For each good segment, a heavy position is (arbitrarily) selected and the representative is contained in S. Thus, there is a bijection between S and the set of the good segments. For the case of Figure 1: $H = \{a_2\}$ and $S = \{a_1, a_4\}$ and $K = A_5$. The resulting set R would be of size $(9+10)-(5-1-2-1) = 10$.

Proposition 3. *The metric dimension of G is equal to $|A \cup B| - k - g - h - |K|$, and R is a minimum resolving set for G.*

Proof. Clearly, $|R| = |A \cup B| - |\{b_1, \ldots, b_k\}| - |H| - |S| - |K| = |A \cup B| - k - h - g - |K|$. So, if R is a resolving set for G, then R is a minimum resolving set for G due to Lemma 4, and the result follows.

We verify that R is indeed a resolving set for G.

Claim. $R \neq \emptyset$, and $\mathfrak{Eq}(R) \sqsubseteq \{A, B\}$.

Proof. If $R = \emptyset$, then $A \subseteq H \cup S \cup K \subseteq \{a_1, \ldots, a_k\}$ and $B \subseteq \{b_1, \ldots, b_k\}$. The definition of high and heavy positions implies $H \cup S = \emptyset$, so that $A \subseteq K \subseteq \{a_k\}$ follows, and therefore $k = 1$. Since $|A \cup B| \geq 3$ and $B = B_1$, $|B_1| \geq 2$ follows, and $B_1 \not\subseteq \{b_1\}$ yields a contradiction.

The second part of the claim follows from the distance properties of connected bipartite graphs. ♢

As a consequence of the result of the claim, it remains to provide a resolving vertex from R for every vertex pair from $H \cup S \cup K$ and from $\{b_1, \ldots, b_k\}$.

We consider $H \cup S \cup K$. Recall from the definition of R that $|B_j| \geq 2$ implies $B_j \cap R \neq \emptyset$. Let $1 \leq s < t \leq k$, and assume $a_s, a_t \in H \cup S \cup K$. Since $s < k$, $a_s \notin K$, and thus, $a_s \in H \cup S$, and $|A_s| \geq 2$. If $a_t \in H$, then $|B_t| \geq 2$, and each vertex in $R \cap B_t$ resolves a_s and a_t. If $a_t \in S$, then t is a heavy position in a good segment $[p, q]$ of G. Since only one heavy position in $[p, q]$ is selected and $|A_s| \geq 2$, $s \leq p-2$ follows. This means $p \geq 3$ and $|B_{p-1}| \geq 2$, and each vertex in $R \cap B_{p-1}$ resolves a_s and a_t. Finally, assume $a_t \in K$. This means $A_k = \{a_k\}$, and $[p, k]$ for every $1 \leq p \leq k$ is not a good segment of G. If $|B_k| \geq 2$, then each vertex in $R \cap B_k$ resolves a_s and a_k. Otherwise, if $|B_k| = 1$, then there is some p such that $[p, k]$ is a segment of G. Since $[p, k]$ is not a good segment of G, there is no heavy position in $[p, k]$, so that $s < p-2$ and $|B_{p-1}| \geq 2$ follows, and R has a resolving vertex for a_s and a_k.

We consider $\{b_1, \ldots, b_k\}$. Let $1 \leq s < t \leq k$. If $|A_s| \geq 2$, then R has a vertex in A_s that resolves b_s and b_t. Otherwise, $|A_s| = 1$, i.e., $A_s = \{a_s\}$. Observe that s is not a high position and that s is not a heavy position in any good segment. Moreover, $a_s \notin K$ because of $s < k$. Thus, $a_s \in R$, and a_s resolves b_s and b_t. □

We are ready to present our main result.

Theorem 1. *Given a chain graph G, a minimum resolving set for G can be constructed in linear time. As a consequence, the* Metric Dimension *problem can be solved in linear time on chain graphs.*

Proof. Due to Proposition 1 and the subsequent discussion, it suffices to consider connected chain graphs on at least three vertices, and we can apply Proposition 3.

The twin classes of a given chain graph G, namely $A_1, \dots, A_k, B_1, \dots, B_k$, can be computed in linear time [15]. The segments, the high positions, the heavy positions of the segments, and K can be determined in $\mathcal{O}(k)$ time from the cardinality sequences. So, a minimum resolving set R as in Proposition 3 can be constructed in linear time. □

Our second result, an exact formula for the metric dimension of special chain graphs, is a direct consequence of Proposition 3.

Theorem 2. *Let $G = (A, B, E)$ be a connected chain graph on at least four vertices. If each twin class of G is of size 1, then $\dim(G) = \frac{|A \cup B|}{2} - 1$.*

Proof. We apply Proposition 3, and it suffices to observe: $|A \cup B| = 2k$, and $g = h = 0$ and $|K| = 1$, so that $\dim(G) = |A \cup B| - k - g - h - |K| = k - 1$. □

5 Variants of metric dimension

Several variants of metric dimension are considered in the literature. Chartrand, Saenpholphat, and Zhang define the *independent resolving number* [7], which asks for the smallest size of a resolving set that is also an independent set, if such a resolving set exists. Other variants of metric dimension vary the metric itself. As an example, the *adjacency metric dimension* of a graph G, $\dim_A(G)$, asks for a set of vertices of G of smallest size that resolves the vertex pairs of G with respect to $\mathrm{dist}_G^{\leq 2}(x, y) = \min\{\mathrm{dist}_G(x, y), 2\}$; we refer to [13,19] for more information.

The concept of a locating-dominating set combines the two above variations. A set of vertices of a graph G is *locating-dominating* if it is a dominating set of G and an adjacency-resolving set for G at the same time [5,23]. The associated parameter is the *location-domination number*, and it is denoted as γ_L.

Lemma 5. *For every connected graph G, $\dim(G) \leq \dim_A(G) \leq \gamma_L(G) \leq \dim_A(G) + 1$.*

Proof. The first two inequalities, $\dim(G) \leq \dim_A(G) \leq \gamma_L(G)$, are straightforward consequences of the definitions.

For verifying the third inequality, $\gamma_L(G) \leq \dim_A(G) + 1$, let R be an adjacency-resolving set for G. If G has a vertex pair u, v where $u \neq v$ and no neighbor of u and v is contained in R, then no vertex in R adjacency-resolves u, v, which is not possible. So, at most one vertex w of G has no neighbor in R, and R itself or $R \cup \{w\}$ is a dominating set of G and adjacency-resolving for G at the same time. □

The example of induced paths of arbitrary length shows that adjacency metric dimension may be much larger than metric dimension. If we restrict to chain graphs, this large gap is bounded. We verify this bound informally. Let $G = (A, B, E)$ be a connected chain graph and let R be a resolving set for G. If R does not adjacency-resolve a vertex pair u, v, then $\mathrm{dist}_G^{\leq 2}(x, u) = \mathrm{dist}_G^{\leq 2}(x, v)$ for every $x \in R$ but $\mathrm{dist}_G(y, u) \neq \mathrm{dist}_G(y, v)$ for some $y \in R$.

Thus, and by symmetry, we can assume $1 < \text{dist}_G(y,u) < \text{dist}_G(y,v) < 4$, and it follows: $u \in A$ if and only if $v \in B$. It suffices to increment R by a vertex from $B_1 \cup A_k$ (with the meanings as in Figure 1), and $\dim_A(G) \leq \dim(G) + 1$ follows.

We reconsider our restriction to connected chain graphs without twins, that admits exact formulas for adjacency metric dimension and the location-domination number.

Corollary 1. *Let $G = (A, B, E)$ be a connected chain graph on at least six vertices whose twin classes are of size 1. Then,* $\dim_A(G) = \frac{|A \cup B|}{2} - 1$ *and* $\gamma_L(G) = \frac{|A \cup B|}{2}$.

Proof. Let $a_1, \ldots, a_k, b_1, \ldots, b_k$ be the representatives of the twin classes of G, as they were chosen at the beginning of Section 3, and where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. Note that $\{a_1, \ldots, a_{k-1}\}$ is the minimum resolving set of G considered in Proposition 3. It is an exercise to verify that also $\{a_2, \ldots, a_k\}$ is a resolving set for G, which proves $\dim_A(G) \leq k - 1 = \frac{|A \cup B|}{2} - 1$.

For the location-domination number, observe that $\{a_1, \ldots, a_k\}$ is a locating-dominating set for G, proving $\gamma_L(G) \leq k$, and a counting argument of the following form proves optimality (recall the notations from Section 3): for R a locating-dominating set for G, if $R \cap A_1 = \emptyset$, then $B_1 \subseteq R$, and for $1 \leq s < t \leq k$, if $R \cap (A_s \cup A_t) = \emptyset$, then $R \cap B_{s+1,\ldots,t} \neq \emptyset$. □

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